Chapter 6 Linear Transformations and Matrices:

6.1 Definition and Examples:

Definition 1: Let V and W be vector spaces. A function $L: V \longrightarrow W$ is called a linear transformation of V into W if

1) $L(u+w) = L(u) + L(w)$ for every u and v in V.

2) $L(cu) = c\,L(u)$ for any u in V, and c any real number.

If $V = W$, the linear transformation $L: V \longrightarrow W$ is called a linear operator on V.

Example 2: Let A be an $m \times n$ matrix. We define a function $L: R^n \longrightarrow R^m$ defined by

$$L(u) = Au.$$

We now show that this function (matrix transformation) by verifying that properties 1) and 2) in Definition 1 hold.

If u and v are vectors in R^n, then

$$L(u+v) = A(u+v) = Au + Av = L(u) + L(v).$$

Moreover, if c is a scalar, then

$$L(cu) = A(cu) = c(Au) = c\,L(u).$$

Hence, every matrix transformation is a linear transformation.

Example 3: Let $L: R^3 \longrightarrow R^3$ be defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 1 \\ 2u_2 \\ u_3 \end{bmatrix}.$$

To determine whether L is a linear transformation, let

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Then

$$L(u+v) = L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = L\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} (u_1 + v_1) + 1 \\ 2(u_2 + v_2) \\ u_3 + v_3 \end{bmatrix}.$$

On the other hand,

$$L(u) + L(v) = \begin{bmatrix} u_1 + 1 \\ 2u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 + 1 \\ 2v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} (u_1 + v_1) + 2 \\ 2(u_2 + v_2) \\ u_3 + v_3 \end{bmatrix}.$$

Letting $u_1 = 1$, $u_2 = 3$, $u_3 = -2$, $v_1 = 2$, $v_2 = 4$, and $v_3 = 1$, we see that $L(u+v) \neq L(u) + L(v)$. Hence we conclude that the function L is not a linear transformation.

Example 4: Let $L: P_1 \longrightarrow P_2$ be defined by

$$L(p(t)) = t\,p(t).$$

Show that L is a linear transformation.

Solution: Let $p(t)$ and $q(t)$ be vectors in P_1 and let c be a scalar. Then

$$\begin{aligned} L(p(t)+q(t)) &= t\,(p(t)+q(t)) \\ &= t\,p(t) + t\,q(t) \\ &= L(p(t)) + L(q(t)). \end{aligned}$$

and

$$\begin{aligned} L(cp(t)) &= t\,(c\,p(t)) \\ &= c\,(t\,p(t)) \\ &= c\,L(p(t)). \end{aligned}$$

Hence L is a linear transformation.

We now develop some general properties of linear transformations.

Theorem 7: Let $L: V \longrightarrow W$ be a linear transformation. Then

① $L(0_V) = 0_W$ ② $L(u - v) = L(u) - L(v)$ $\forall$ u, v in V.

Proof:

1) We have

$$0_V = 0_V + 0_V,$$

So

$$L(0_V) = L(0_V + 0_V)$$

$$= L(0_V) + L(0_V).$$

Adding $-L(0_V)$ to both sides, we obtain

$$L(0_V) = 0_W.$$

2) $L(u - v) = L(u + (-1)v) = L(u) + L(-1 \cdot v)$

$$= L(u) + (-1)L(v)$$

$$= L(u) - L(v).$$

6.2 Kernel and Range of a Linear Transformation:

Definition 1: A linear transformation $L: V \longrightarrow W$ is called one-to-one if it is a one-to-one function; that is, if $v_1 \neq v_2$ implies that $L(v_1) \neq L(v_2)$. An equivalent statement is that L is one-to-one if $L(v_1) = L(v_2)$ implies that $v_1 = v_2$.

Example 2: Let $L: R^2 \longrightarrow R^2$ be defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + u_2 \\ u_1 - u_2 \end{bmatrix}.$$

To determine whether L is one-to-one, we get

$$v_1 = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Then, if $L(v_1) = L(v_2)$, we have

$$u_1 + u_2 = v_1 + v_2$$
$$u_1 - u_2 = v_1 - v_2.$$

Adding these equations, we obtain $2u_1 = 2v_1$, or $u_1 = v_1$, which implies that $u_2 = v_2$. Hence $v_1 = v_2$, and L is one-to-one.

Example 3: Let $L: R^3 \longrightarrow R^2$ be the linear transformation defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Since $L\left(\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}\right) = L\left(\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}\right)$, yet $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, we conclude that L is not one-to-one.

* We shall now develop some more efficient ways of determining whether a linear transformation is one-to-one.

Definition 4: Let $L: V \longrightarrow W$ be a linear transformation of a vector space V into a vector space W. The kernel of L, $\ker L$, is the subset of V consisting of all elements v of V such that $L(v) = 0_W$.

Remark 5: We observe that Theorem assures us that $\ker L$ is never an empty set, because if $L: V \longrightarrow W$ is a linear transformation, then 0_V is in $\ker L$.

Example 6: Let $L: R^3 \longrightarrow R^2$ be as defined in Example 3. The vector $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ is in $\ker L$, since $L\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. However, the vector $\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$ is not in $\ker L$, since $L\left(\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

To find $\ker L$, we must determine all v in R^3 so that $L(v) = 0_{R^2}$. That is, we seek $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ so that

$$L(v) = L\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0_{R^2}.$$

However, $L(v) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Thus $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $v_1 = 0$, $v_2 = 0$, and v_3 can be any real number. Hence $\ker L$ consists of all vectors in R^3 of the form $\begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}$, where a is any real number.

Theorem 7: Let $L: V \longrightarrow W$ be a linear transformation of a vector space V into a vector space W. Then

1) $\ker L$ is a subspace of V.
2) L is one-to-one if and only if $\ker L = \{0_V\}$.

Proof: 1) We show that if v and w are in $\ker L$, then so are $v+w$ and cv for any real number c. If v and w are in $\ker L$, then $L(v) = 0_W$, and $L(w) = 0_W$. Then, since L is a linear transformation,

$$L(v+w) = L(v) + L(w) = 0_W + 0_W = 0_W.$$

Thus $v+w$ is in $\ker L$. Also,

$$L(cv) = c\,L(v) = c\,0_W = 0_W,$$

so cv is in $\ker L$. Hence $\ker L$ is a subspace of V.

2) Let L be one-to-one function. We shall that $\ker L = \{0_V\}$. Let v be in $\ker L$. Then $L(v) = 0_W$. Also, we already know that $L(0_V) = 0_W$. Then $L(v) = L(0_V)$. Since L is one-to-one, we conclude that $v = 0_V$. Hence $\ker L = \{0_V\}$.

Conversely, suppose that $\ker L = \{0_V\}$. We wish to show that L is one-to-one. Let $L(v_1) = L(v_2)$ for v_1 and v_2 in V. Then

$$L(v_1) - L(v_2) = 0_W,$$

So that $L(v_1 - v_2) = 0_W$. This means that $v_1 - v_2$ is in $\ker L$, so $v_1 - v_2 = 0_V$. Hence $v_1 = v_2$, and L is one-to-one.

Remark 8: Note that we can also state Theorem 7(2) as follows:

L is one-to-one if and only if $\dim \ker L = 0$.

Example 10: Let $L: P_2 \longrightarrow R$ be the linear transformation defined by

$$L(at^2 + bt + c) = \int_0^1 (at^2 + bt + c)\, dt.$$

1) Find $\ker L$.

2) Find $\dim \ker L$.

3) Is L one-to-one.

Solution:

1) To find $\ker L$, we seek an element $v = at^2 + bt + c$ in P_2 such that $L(v) = L(at^2 + bt + c) = 0_R = 0$. Now

$$L(v) = \frac{at^3}{3} + \frac{bt^2}{2} + ct \Big|_0^1 = \frac{a}{3} + \frac{b}{2} + c.$$

Thus $c = -\frac{a}{3} - \frac{b}{2}$. Then $\ker L$ consists of all polynomials in P_2 of the form $at^2 + bt + \left(-\frac{a}{3} - \frac{b}{2}\right)$, for a and b any real numbers.

2) To find the dimension of $\ker L$, we obtain a basis for $\ker L$. Any vector in $\ker L$ can be written as

$$at^2 + bt + \left(-\frac{a}{3} - \frac{b}{2}\right) = a\left(t^2 - \frac{1}{3}\right) + b\left(t - \frac{1}{2}\right).$$

Thus the elements $t^2 - \frac{1}{3}$ and $t - \frac{1}{2}$ in P_2 span $\ker L$. Now, these elements are also linearly independent. Thus $\left\{t^2 - \frac{1}{3}, t - \frac{1}{2}\right\}$ is a basis for $\ker L$, and $\dim \ker L = 2$.

3) Since $\dim \ker L = 2$, L is not one-to-one.